Types so far:

- Separable
- Linear
- Exact

Warm-up

Solve.

- $x\frac{dy}{dx}+(3x+1)y=e^{-3x}$

- $\left(\frac{1}{1+y^2}+\cos x-2xy\right)\frac{dy}{dx}=y(y+\sin x)$, $y(0)=1$

- $x(1+y^2)^{1/2}dx=y(1+x^2)^{1/2}dy$

- $x\boxed{\frac{dy}{dx}}+(3x+1)\boxed{y}=e^{-3x}$ linear

assume $x>0$

Std. form: $\frac{dy}{dx}+\left(3+\frac{1}{x}\right)y=\frac{e^{-3x}}{x}$

$\mu=e^{\int(3+\frac{1}{x})dx}=e^{3x+\ln x}=e^{3x}e^{\ln x}=xe^{3x}$

$\frac{d}{dx}(\mu y)=\frac{d}{dx}(xe^{3x}y)=1\Rightarrow xe^{3x}y=x+C$

$y=e^{-3x}+\frac{c}{x}e^{-3x}$ $I:(0,\infty)$

$$\left(\frac{1}{1+y^2} + \cos x - 2xy\right)\frac{dy}{dx} = y(y+\sin x), \quad y(0)=1$$

$f(x,y)$

$\frac{\partial}{\partial x}$ M N $\frac{\partial}{\partial y}$

$\cancel{M_y \stackrel{?}{=} N_x}$

Exact: $M\,dx + N\,dy = 0$

$$(y^2 + y\sin x)\,dx + \left(2xy - \frac{1}{1+y^2} - \cos x\right)dy = 0$$

$$M_y = 2y + \sin x \qquad N_x = 2y + \sin x$$

$$f(x,y) = \int M\,dx = \int (y^2 + y\sin x)\,dx = xy^2 - y\cos x + h(y)$$

$f_y = N$: $\cancel{2xy} - \cancel{\cos x} + h'(y) = \cancel{2xy} - \frac{1}{1+y^2} - \cancel{\cos x}$

$$h'(y) = -\frac{1}{1+y^2} = -\tan^{-1} y + k$$

$\cancel{f(x,y)} = xy^2 - y\cos x - \tan^{-1} y = C$

Form: $f(x,y) = C$

$y(0)=1 \Rightarrow$ when $x=0$, $y=1$

$$0 - 1 - \tan^{-1} 1 = C \Rightarrow C = -1 - \frac{\pi}{4}$$

$$xy^2 - y\cos x - \tan^{-1} y = -1 - \frac{\pi}{4}$$

$$x^2+2xy+y^2=C$$
$$f_x\,dx+f_y\,dy=0$$

$$x^2+y^2=C$$
$$2x+2y\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=-\frac{x}{y}$$

$$\bullet\ 2x(1+y^2)^{1/2}dx=2y(1+x^2)^{1/2}dy$$

Separable: $\displaystyle\int 2x(1+x^2)^{-1/2}dx=\int 2y(1+y^2)^{-1/2}dy$

$u=1+x^2$
$du=2x\,dx$

$$\frac{2\sqrt{1+x^2}}{2}+\frac{C_1}{2}=\frac{2\sqrt{1+y^2}}{2}$$

$$1+y^2=\left(\sqrt{1+x^2}+c\right)^2$$

2.5 - Solutions by Substitutions

We will use substitutions to solve DEs (and IVPs) of the following types.

$\left(y^2 + xy\right)\, dx + x^2\, dy = 0$

hom homog

An equation of the form $M(x, y)\, dx + N(x, y)\, dy = 0$ is a homogeneous equation, if the coeicient functions possess the property $f(tx, ty) = t^\alpha f(x, y)$ (with the same α for both functions).

$t^2 M$

$M(x,y) = y^2 + xy \Rightarrow M(tx, ty) = t^2y^2 + t^2xy = t^2(y^2+xy)$

$N(tx, ty) = t^2x^2 \Rightarrow M(tx,ty)dx + N(tx,ty)dy = 0$

becomes $t^2 M(x,y)dx + t^2 N(x,y) = 0$

Substitution:

$y = ux$ OR $x = vy$

$dy = xdu + udx$ $dx = ydv + vdy$

$\dfrac{dy}{dx} - y = e^x y^2$ $n = 2$

The DE $\dfrac{dy}{dx} + P(x)y = f(x)y^n$ is called Bernoulli's equation.

Substitution: $u = y^{1-n}$

Here, $u = y^{-1} \Rightarrow y = \frac{1}{u} = u^{-1}$

$\dfrac{dy}{dx} = -\dfrac{1}{u^2}\dfrac{du}{dx}$

The equation $\dfrac{dy}{dx} = f(Ax + By + C)$ can be converted to a separable equation using the substitution $u = Ax + By + C$.

$$u = Ax + By + C \Rightarrow \frac{du}{dx} = A + B\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{B}\left(\frac{du}{dx} - A\right)$$

$$\frac{dy}{dx} = \frac{3x + 2y}{3x + 2y + 2}$$

Example: Solve the given di□erential by using an appropriate substitution.

$(y^2 + xy)\ dx + x^2\ dy = 0$ homogeneous

$$y = ux \Rightarrow dy = x\,du + u\,dx$$

$$(u^2\cancel{x^2} + u\cancel{x^2})dx + \cancel{x^2}(x\,du + u\,dx) = 0$$

$$u^2dx + u\,dx + x\,du + u\,dx = 0$$

$$(u^2 + 2u)dx + x\,du = 0 \qquad \text{separable}$$

$$\int \frac{1}{x}dx + \int \frac{1}{u^2+2u}du = \int 0\,dx$$

Partial fraction decomposition:

$$\frac{1}{u(u+2)} = \frac{A}{u} + \frac{B}{u+2} \Rightarrow 1 = A(u+2) + Bu$$

$$u = 0 \Rightarrow A = \frac{1}{2}$$

$$u = -2 \Rightarrow B = -\frac{1}{2}$$

$$\int \frac{1}{x}dx + \int \left(\frac{1/2}{u} - \frac{1/2}{u+2}\right)du = \int 0\,dx$$

$$\ln|x| + \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2| = \ln|c|$$

$$e^{\ln\left(|x|\sqrt{\left|\frac{u}{u+2}\right|}\right)} = e^{\ln|c_1|}$$

$$|x|\sqrt{\left|\frac{u}{u+2}\right|} = |c_1|$$

$$x^2 \frac{u}{u+2} = c$$

But $y = ux$

$u = \frac{y}{x}$

$$x^2 \frac{y/x}{y/x+2} = c \quad \text{OR} \quad \frac{x^2 y}{y+2x} = c$$

$$\boxed{x^2 y = c(y+2x)}$$

$x^2 y = cy + 2cx$

$(x^2 - c)\, y = 2cx \Rightarrow y = \frac{2cx}{x^2-c}$

OR

Example: $\frac{dy}{dx} - y = e^x y^2$

linear

not linear

Bernoulli w/ $n = 2$

$$u = y^{1-2} \Rightarrow u = y^{-1} \Rightarrow y = u^{-1}$$

$$\frac{dy}{dx} = -\frac{1}{u^2}\frac{du}{dx}$$

$$-\frac{1}{u^2}\boxed{\frac{du}{dx}} - \frac{1}{u} = e^x \frac{1}{u^2}$$

$$\frac{du}{dx} + u = -e^x \quad \text{linear}$$

$$\mu = e^{\int 1 dx} = e^x \Rightarrow \frac{d}{dx}(e^x u) = -e^{2x}$$

$$e^x u = -\frac{1}{2}e^{2x} + c_1$$

$$\frac{1}{y} = u = -\frac{1}{2}e^x + c_1 e^{-x} = \frac{-e^{2x}}{2e^x} + \frac{2c_1}{2e^x}$$

$$\frac{1}{y} = \frac{c - e^{2x}}{2e^x} \Rightarrow \boxed{y = \frac{2e^x}{c - e^{2x}}}$$

Example: $3\left(1+t^2\right)\frac{dy}{dt} = 2ty\left(y^3+1\right)$

$$3(1+t^2)\frac{dy}{dt} = 2ty^4 + 2ty$$

Want: $\frac{dy}{dt} + P(t)\,y = g(t)\,y^n$

$$\frac{dy}{dt} - \frac{2t}{3(1+t^2)}y = \frac{2t}{3(1+t^2)}y^4$$

Bernoulli w/ $n = 4$ so $u = (y^{-3})^{(-1/3)} = \frac{1}{y^3}$

$$\boxed{y = u^{-1/3}} \Rightarrow \frac{dy}{dt} = -\frac{1}{3}u^{-4/3}\frac{du}{dt}$$

$$+\frac{1}{\cancel{3}}u^{-4/3}\boxed{\frac{du}{dt}} + \frac{2t}{\cancel{3}(1+t^2)}u^{-1/3} = -\frac{2t}{\cancel{3}(1+t^2)}u^{-4/3}$$

$$\frac{du}{dt} + \frac{2t}{1+t^2}\boxed{u} = -\frac{2t}{1+t^2}$$

$$\mu = e^{\int \frac{2t}{1+t^2}dt} = 1+t^2$$

$$\frac{d}{dt}\left[(1+t^2)u\right] = -2t$$

$$(1+t^2)u = -t^2 + C$$

$$u = \frac{C-t^2}{1+t^2} \qquad u = y^{-3}$$

$$y^3 = \frac{1+t^2}{C-t^2} \quad \underline{\text{OR}} \quad y = \left(\frac{1+t^2}{C-t^2}\right)^{1/3}$$

Example: Solve the given initial-value problem.

$$\boxed{\frac{dy}{dx}} = \frac{3x+2y}{3x+2y+2}, \qquad y(-1) = -1$$

$u = 3x + 2y$

$\frac{du}{dx} = 3 + 2\boxed{\frac{dy}{dx}}$

$\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx} - \frac{3}{2}$

$\frac{1}{2}\frac{du}{dx} - \frac{3}{2} = \frac{u}{u+2}$

$\frac{du}{dx} = \frac{2u}{u+2} + 3\frac{u+2}{u+2} = \frac{5u+6}{u+2}$

$\frac{u+2}{5u+6}\,du = dx$

$5u+6\ \overline{)\,u+2\,}$ $\quad \frac{1}{5} + \frac{4/5}{5u+6}$

$-u - \frac{6}{5}$

$\left(\frac{1}{5} + \frac{1}{5}\frac{4/5 \cdot 5}{5u+6}\right)du = dx$

$t = 5u+6 \Rightarrow dt = 5\,du$

$\left(\frac{1}{5} + \frac{4}{25}\,\frac{5}{5u+6}\right)du = dx$

$\frac{1}{5}u + \frac{4}{25}\ln|5u+6| = x + C_1$ $\qquad C = 5C_1$

$u = 3x+2y$ so $\quad 3x+2y + \frac{4}{5}\ln|15x+10y+6| = 5x + C$

$2y - 2x + \frac{4}{5}\ln|15x+10y+6| = C$